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Cylindrical equilibrium shapes of fluid membranes

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Abstract
Within the framework of the well-known curvature models, a fluid lipid bilayer membrane is regarded as a surface embedded in the three-dimensional Euclidean space whose equilibrium shapes are described in terms of its mean and Gaussian curvatures by the so-called membrane shape equation. In the present paper, all solutions to this equation determining cylindrical membrane shapes are found and presented, together with the expressions for the corresponding position vectors, in explicit analytic form. The necessary and sufficient conditions for such a surface to be closed are derived and several sufficient conditions for its directrix to be simple or self-intersecting are given.

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1. Introduction

By a fluid membrane in this paper we assume a membrane formed in an aqueous solution by a bilayer of lipid molecules, which are in a fluid state, i.e., the molecules can move freely within the monolayer they belong to. The structure of the bilayer is such that the hydrophobic tails of the lipid molecules situated in different monolayers face one another to form a semipermeable core, while their hydrophilic heads face the aqueous solutions on either side of the membrane. It is known that the lipid bilayer is the main structural component of all biological membranes, the closed lipid bilayer membranes (vesicles) being thought of as the simplest model systems for studying basic physical properties of the more complex biological cells.

The foundation of the current theoretical understanding of the vesicle shapes (see, e.g., [20, 30, 31]) can be traced more than thirty years back to the works by Canham [7] and Helfrich [14], in which the so-called curvature models have been introduced. In these models, the vesicle's membrane is regarded as a two-dimensional surface $\mathcal{S}$ embedded in the three-dimensional Euclidean space $\mathbb{R}^3$ and assumed to exhibit purely elastic behaviour described by

its mean H and Gaussian K curvatures and two material constants associated with the bending rigidity of the membrane.

In the model proposed by Helfrich [14], currently referred to as the spontaneous-curvature model, the equilibrium shapes of the vesicles are determined by the extremals of the curvature (shape) energy

$$\mathcal{F}_c = \frac{k_c}{2}\int_{\mathcal{S}}(2H + \mathbb{h})^2\,\mathrm{d}A + k_G\int_{\mathcal{S}} K\,\mathrm{d}A$$

under the constraints of fixed enclosed volume V and total membrane area A. This scheme yields the functional

$$\mathcal{F} = \frac{k_c}{2}\int_{\mathcal{S}}(2H + \mathbb{h})^2\,\mathrm{d}A + k_G\int_{\mathcal{S}} K\,\mathrm{d}A + \lambda\int_{\mathcal{S}}\mathrm{d}A + p\int \mathrm{d}V. \tag{1}$$

Here k_c and k_G are real constants representing the bending and Gaussian rigidity of the membrane, $\mathbb{h}$ is the spontaneous curvature (a constant introduced by Helfrich to reflect the asymmetry of the membrane or its environment), p and λ are the Lagrange multipliers (another two real constants) corresponding to the constraints of enclosed volume and total area, respectively, whose physical meaning is as follows: p represents the pressure difference between the outer and inner sides of the membrane, while λ is interpreted as a tensile stress or a chemical potential (see, e.g., [5]). The Euler–Lagrange equation corresponding to the functional (1) reads

$$2k_c\Delta H + k_c(2H + \mathbb{h})(2H^2 - \mathbb{h}H - 2K) - 2\lambda H + p = 0, \tag{2}$$

where Δ is the Laplace–Beltrami operator on the surface $\mathcal{S}$. This equation, derived in [28], is often referred to as the membrane shape equation. It is worth noting that the second term in the curvature energy $\mathcal{F}_c$ does not affect equation (2) since its contribution to the overall Lagrangian density is a total divergence as follows from the Liouville's form of Gauss's Theorema Egregium (see, e.g., [10]). Actually, for closed membranes without edges the integral over the Gaussian curvature K is a topological invariant by virtue of the Gauss–Bonnet theorem and therefore it may be disregarded until the topology of the membrane remains unchanged. This term, however, plays an important role when topological phase transitions are considered (see, e.g., [3, 24]) as well as in the theory of fluid membranes with free edges (cf [9, 36, 37]).

Later on, other two curvature models have been developed. The first of them is the so-called bilayer-couple model suggested by Svetina and Žekš in [33] on the ground of the bilayer-couple hypothesis [32] and the related work [34]. The second one is known as the area-difference-elasticity model [4, 22, 42]. For the purposes of the present paper, however, it is important to underline that all the curvature models mentioned above lead to the same set of stationary shapes, determined locally by equation (2), since they differ only in global energy terms (see [20, 22, 33]). Of course, the constants involved in this equation have different meanings in different models.

For more than three decades, the study of the equilibrium shapes of the vesicles has attracted much attention nevertheless to the best of our knowledge only a few analytic solutions to equation (2) have been reported. These are the solutions determining: spheres and circular cylinders [28], Clifford tori [29] and toroidal shapes (cf [20], chapter 8, section 5), circular biconcave discoids [25], Delaunay surfaces [23, 26], nodoidlike and unduloidlike shapes [26], several types of surfaces with constant squared mean curvature density and Willmore surfaces (cf [16, 38, 43]) as well as cylindrical surfaces corresponding to $p \neq 0$ [39, 40] and $p = 0$. The latter case is exceptional since it coincides with the prominent Euler's elastica whose typical equilibrium shapes are known for a long time [21]. This problem has been studied for more than three centuries in various contexts, see, e.g., the recent papers [6, 15, 18, 19, 45].

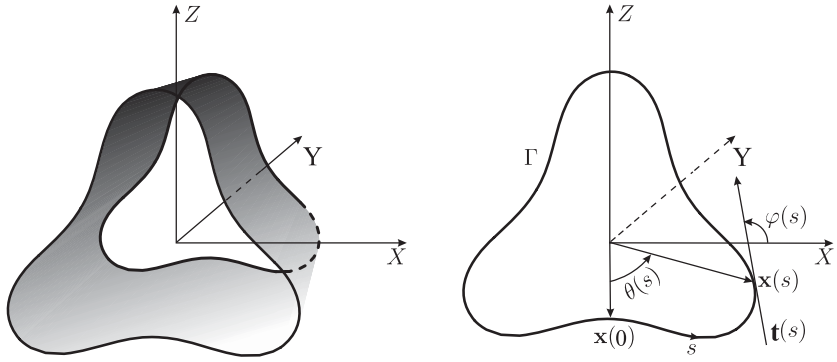

Figure 1. A slice of the infinite generalized cylinder (left) and its intersection with the plane $Y \equiv 0$ (right). Here, $\mathbf{t}(s)$, $\varphi(s)$ and $\theta(s)$ are the tangent vector, slope angle and the angle between the position vectors $\mathbf{x}(0)$ and $\mathbf{x}(s)$, respectively.

The main goal of the present paper is to present in analytic form all solutions to the membrane shape equation (2) determining cylindrical surfaces as well as to provide explicit expressions for the corresponding position vectors. In a sense, it might be thought of as a completion, from analytic point of view, of the works by Arreaga *et al* [2] and Capovilla *et al* [8] where a purely geometric construction of the equilibrium shapes of closed planar loops subject to the constraints of fixed length and enclosed area is presented.

2. Cylindrical equilibrium shapes

For cylindrical surfaces in $\mathbb{R}^3$ whose directrices are plane curves Γ of curvature $\kappa(s)$ parametrized by their arclength s, the corresponding generatrices being perpendicular to the plane the directrices Γ lie in (see figure 1), the general shape equation (2) simplifies and reduces to the single ordinary differential equation

$$2\frac{\mathrm{d}^2\kappa(s)}{\mathrm{d}s^2} + \kappa^3(s) - \mu\kappa(s) - \sigma = 0, \tag{3}$$

where

$$\mu = \mathbf{h}^2 + \frac{2\lambda}{k_c}, \qquad \sigma = -\frac{2p}{k_c}.$$

Indeed, using the standard formulae from the textbooks on classical differential geometry (see, e.g., [10, 27]) one can easily find that for a cylindrical surface parametrized in the above way $H = (1/2)\kappa(s)$ and $\Delta H = (1/2)\,\mathrm{d}^2\kappa(s)/\mathrm{d}s^2$ in addition to the relation $K = 0$. Substituting the latter expressions into equation (2) one immediately obtains equation (3).

In what follows, we are interested in real-valued solutions $\kappa(s) \neq \text{const}$ of equation (3) possessing smooth derivatives. Once such a solution is known, it is possible to recover the embedding $\mathbf{x}(s) = (x(s), z(s)) \in \mathbb{R}^2$ of the corresponding directrix Γ in the plane $\mathbb{R}^2$ (up to a rigid motion) in the standard manner. First, recall that the unit tangent $\mathbf{t}(s) = (\mathrm{d}x(s)/\mathrm{d}s, \mathrm{d}z(s)/\mathrm{d}s)$ and normal $\mathbf{n}(s) = (-\mathrm{d}z(s)/\mathrm{d}s, \mathrm{d}x(s)/\mathrm{d}s)$ vectors to the curve Γ are related to the curvature $\kappa(s)$ through Frenet–Serret formulae [10, 27]

$$\frac{\mathrm{d}\mathbf{t}(s)}{\mathrm{d}s} = \kappa(s)\mathbf{n}(s), \qquad \frac{\mathrm{d}\mathbf{n}(s)}{\mathrm{d}s} = -\kappa(s)\mathbf{t}(s). \tag{4}$$

Consequently, in terms of the slope angle $\varphi(s)$ of the curve Γ (see figure 1), one has

$$\kappa(s) = \frac{\mathrm{d}\varphi(s)}{\mathrm{d}s}, \qquad \frac{\mathrm{d}x(s)}{\mathrm{d}s} = \cos(\varphi(s)), \qquad \frac{\mathrm{d}z(s)}{\mathrm{d}s} = \sin(\varphi(s)) \tag{5}$$

and hence, the parametric equations of the curve Γ can be expressed by quadratures

$$x(s) = \int \cos(\varphi(s))\,\mathrm{d}s, \qquad z(s) = \int \sin(\varphi(s))\,\mathrm{d}s, \tag{6}$$

where

$$\varphi(s) = \int \kappa(s)\,\mathrm{d}s. \tag{7}$$

Thus, the first problem to solve on the way to determining the cylindrical equilibrium shapes of the fluid membranes is to find the solutions of equation (3) in analytic form.

Fortunately, equation (3) is integrable by quadrature since it falls into the class of equations describing conservative systems with one degree of freedom [1]. Indeed, (3) can be regarded as the equation of motion of a fictitious particle of unit mass whose kinetic, potential and total energies are

$$T = \frac{1}{2}\left(\frac{\mathrm{d}\kappa}{\mathrm{d}s}\right)^2, \qquad U(\kappa) = \frac{1}{8}\kappa^4 - \frac{1}{4}\mu\kappa^2 - \frac{1}{2}\sigma\kappa, \qquad \mathcal{E} = T + U,$$

respectively. In this setting, κ is interpreted as the displacement of the particle while s plays the role of the time. The total energy $\mathcal{E}$ of this system is conserved and hence

$$\left(\frac{\mathrm{d}\kappa}{\mathrm{d}s}\right)^2 = P(\kappa), \qquad P(\kappa) = 2E - \frac{1}{4}\kappa^4 + \frac{1}{2}\mu\kappa^2 + \sigma\kappa \tag{8}$$

holds on each continuous solution of equation (3), E being the value of its total energy. Therefore, the solution of equation (3) can be reduced to the quadrature

$$s = \int \frac{\mathrm{d}\kappa}{\sqrt{2(E - U(\kappa))}} = \int \frac{\mathrm{d}\kappa}{\sqrt{P(\kappa)}} \tag{9}$$

up to a shift of the independent variable s and change of its sign to the opposite one. Note that equation (3) and its first integral (8) are invariant under the aforementioned transformations of the variable s. Note also that for each solution $\kappa = \kappa(s)$ of equation (3), relation (8) implies the existence of a certain value of the variable s at which $\mathrm{d}\kappa/\mathrm{d}s = 0$ (this matter will be clarified in detail at the beginning of the following section). Without loss of generality, this value may be chosen to be zero due to the translational invariance of equation (3). So, hereafter we will always chose $\mathrm{d}\kappa/\mathrm{d}s = 0$ at $s = 0$.

Moreover, the specific differential structure of equation (3) allows the integration in expressions (6) to be avoided when $\sigma \neq 0$. Below, one can find a simple alternative derivation of this remarkable integrability property, first established in [2] (see also [8, 12]). Actually, a direct computation shows that the following identity holds:

$$\begin{aligned}\left(2\frac{\mathrm{d}^2\kappa(s)}{\mathrm{d}s^2} + \kappa^3(s) - \mu\kappa(s) - \sigma\right)\mathbf{t}(s) + 2\frac{\mathrm{d}\kappa(s)}{\mathrm{d}s}\left(\frac{\mathrm{d}\mathbf{t}(s)}{\mathrm{d}s} - \kappa(s)\mathbf{n}(s)\right)\\ - (\kappa^2(s) - \mu)\left(\frac{\mathrm{d}\mathbf{n}(s)}{\mathrm{d}s} + \kappa(s)\mathbf{t}(s)\right)\\ = \frac{\mathrm{d}}{\mathrm{d}s}\left(2\frac{\mathrm{d}\kappa(s)}{\mathrm{d}s}\mathbf{t}(s) - (\kappa^2(s) - \mu)\mathbf{n}(s) - \sigma\mathbf{x}(s)\right)\end{aligned} \tag{10}$$

and hence, taking into account Frenet–Serret formulae (4), one can represent the position vector $\mathbf{x}(s)$ of a plane curve of curvature $\kappa(s)$ in the form

$$\mathbf{x}(s) = \frac{2}{\sigma}\frac{\mathrm{d}\kappa(s)}{\mathrm{d}s}\mathbf{t}(s) - \frac{1}{\sigma}(\kappa^2(s) - \mu)\mathbf{n}(s) + \mathbf{C},$$

where $\mathbf{C}$ is a constant vector, provided that $\kappa(s)$ is a solution of equation (3) with $\sigma \neq 0$. Then, translating the origin so that $\mathbf{x}\cdot\mathbf{t} = 0$ and $\mathbf{x}\cdot\mathbf{n} = -1/\sigma(\kappa^2 - \mu)$ when $\mathrm{d}\kappa/\mathrm{d}s = 0$, which is always possible, one gets $\mathbf{C} = \mathbf{0}$ and obtains, taking into account the definitions of the tangent and normal vectors as well as the second and the third of relations (5), the following expressions for the components of the position vector in terms of the curvature $\kappa(s)$ and slope angle $\varphi(s)$

$$\begin{aligned} x(s) &= \frac{2}{\sigma}\frac{\mathrm{d}\kappa(s)}{\mathrm{d}s}\cos\varphi(s) + \frac{1}{\sigma}(\kappa^2(s) - \mu)\sin\varphi(s), \\ z(s) &= \frac{2}{\sigma}\frac{\mathrm{d}\kappa(s)}{\mathrm{d}s}\sin\varphi(s) - \frac{1}{\sigma}(\kappa^2(s) - \mu)\cos\varphi(s). \end{aligned} \tag{11}$$

Note, however, that the slope angle $\varphi(s)$ still remains determined implicitly, via an integration (7) of the curvature $\kappa(s)$, which, so far, cannot be accomplished. Note also that formulae (8) and (11) lead to the remarkable relation

$$r^2(s) = \frac{8E + \mu^2}{\sigma^2} + \frac{4\kappa(s)}{\sigma} \tag{12}$$

for the magnitude $r(s) = \sqrt{x^2(s) + z^2(s)}$ of the position vector $\mathbf{x}(s)$ found in [2, 8].

For $\sigma = 0$, the situation is quite similar. In this case, identity (10) implies

$$\frac{\mathrm{d}\kappa(s)}{\mathrm{d}s}\mathbf{t}(s) - \frac{1}{2}(\kappa^2(s) - \mu)\mathbf{n}(s) = \mathbf{C_0}, \tag{13}$$

where $\mathbf{C_0}$ is a constant vector, provided that $\kappa(s)$ is a solution of equation (3) with $\sigma = 0$ and relations (4) hold. Now, choosing $\varphi = \mathrm{d}\kappa/\mathrm{d}s = 0$ at $s = 0$ and taking into account relation (8), the definitions of the tangent and normal vectors as well as the second and the third of relations (5), one can first see that $\mathbf{C_0} = (0, -(\kappa^2(0) - \mu)/2) \neq \mathbf{0}$ and then rewrite equality (13) in the form

$$\cos\varphi(s) = \frac{\kappa^2(s) - \mu}{\kappa^2(0) - \mu}, \qquad \sin\varphi(s) = -\frac{2}{\kappa^2(0) - \mu}\frac{\mathrm{d}\kappa(s)}{\mathrm{d}s}. \tag{14}$$

Consequently, formulae (6) and (14) imply

$$x(s) = \frac{1}{\kappa^2(0) - \mu}\int \kappa^2(s)\,\mathrm{d}s - \frac{\mu s}{\kappa^2(0) - \mu}, \qquad z(s) = -\frac{2\kappa(s)}{\kappa^2(0) - \mu}. \tag{15}$$

Again, one integration remains to be done. This time, the square of the curvature $\kappa(s)$ has to be integrated.

Before proceeding with the derivation of the solutions of equation (3) using the quadrature (9), it should be remarked that these equation have been regarded in a number of papers (see [11, 35, 41, 44]) that have not been mentioned yet since they do not concern directly the problem considered here. Closest to the subject of the present paper are [2, 8] where equation (3) is introduced with the aim to study the equilibria of an elastic loop in the plane, subject to the constraints of fixed length and enclosed area. In the three-dimensional case considered here, each such loop determines a directrix Γ of a cylindrical surface whose mean curvature satisfies the membrane shape equation. In the foregoing two papers, the authors have succeeded in obtaining a purely geometric construction for determination of the curvature of the loop passing through a given point of the plane without using explicit expressions for the solutions of equation (3). Nevertheless, in our opinion, the knowledge of the solutions to this equation in analytic form is of considerable interest for further exploration of the cylindrical equilibrium shapes of the fluid membranes.

3. Explicit analytic solutions

First, it should be remarked that the quadrature (9) can be easily expressed in terms of elliptic integrals [13] or elementary functions by means of the roots of the polynomial $P(\kappa)$ provided that the following observations are taken into account.

Given a solution $\kappa(s)$ of an equation of form (3) with coefficients μ and σ, let E be the value of its total energy. Then, bearing in mind that μ, σ and E are real numbers, it is clear that the corresponding polynomial $P(\kappa)$ has at least two different real roots, otherwise the function $P(\kappa(s))$ could not take different non-negative values, as required by relation (8) and the assumptions concerning the type of solutions considered, since the coefficient at the highest power κ^4 of this polynomial is negative. Thus, in general, only two alternative possibilities have to be considered, namely: (I) the polynomial $P(\kappa)$ has two simple real roots $\alpha, \beta \in \mathbb{R}, \alpha < \beta$, and a pair of complex conjugate roots $\gamma, \delta \in \mathbb{C}, \delta = \bar{\gamma}$; (II) the polynomial $P(\kappa)$ has four simple real roots $\alpha < \beta < \gamma < \delta \in \mathbb{R}$. In the first case, the polynomial $P(\kappa)$ is non-negative in the interval $\alpha \leqslant \kappa \leqslant \beta$, while in the second one, it is nonnegative in the intervals $\alpha \leqslant \kappa \leqslant \beta$ and $\gamma \leqslant \kappa \leqslant \delta$.

It should be noted also that the roots α, β, γ and δ of the polynomial $P(\kappa)$ can be expressed explicitly through its coefficients μ, σ and E and vice versa. Indeed, after some standard algebraic manipulations (cf [17], section 1.8), one can find the following expressions for the roots of the polynomial $P(\kappa)$:

$$-\sqrt{\frac{\omega}{2}} - \sqrt{\mu - \sigma\sqrt{\frac{2}{\omega}} - \frac{\omega}{2}}, \qquad -\sqrt{\frac{\omega}{2}} + \sqrt{\mu - \sigma\sqrt{\frac{2}{\omega}} - \frac{\omega}{2}},$$

$$\sqrt{\frac{\omega}{2}} - \sqrt{\mu + \sigma\sqrt{\frac{2}{\omega}} - \frac{\omega}{2}}, \qquad \sqrt{\frac{\omega}{2}} + \sqrt{\mu + \sigma\sqrt{\frac{2}{\omega}} - \frac{\omega}{2}},$$

where

$$\omega = \frac{\left[\mu + \sqrt[3]{3(3^2\sigma^2 + \sqrt{\chi}) - \mu(\mu^2 + 2^3 3^2 E)}\right]^2 - 2^3 3E}{3\sqrt[3]{3(3^2\sigma^2 + \sqrt{\chi}) - \mu(\mu^2 + 2^3 3^2 E)}},$$

$$\chi = 3\{2^3 E[(\mu^2 + 8E)^2 - 3^2 2\mu\sigma^2] - \sigma^2(2\mu^3 - 3^3\sigma^2)\}.$$

Then, one can denote properly each of the above expressions in accordance with the notation introduced in cases (I) and (II), respectively. Simultaneously, by Vieta's formulae one obtains

$$\alpha + \beta + \gamma + \delta = 0 \tag{16}$$

due to the absence of a term with κ^3 in the polynomial $P(\kappa)$, and consequently

$$\mu = \tfrac{1}{2}(\alpha^2 + \beta^2 + \gamma^2 + \alpha\beta + \alpha\gamma + \beta\gamma), \tag{17}$$

$$\sigma = -\tfrac{1}{4}(\alpha + \beta)(\alpha + \gamma)(\beta + \gamma), \tag{18}$$

$$E = \tfrac{1}{8}\alpha\beta\gamma(\alpha + \beta + \gamma). \tag{19}$$

Now, we are in a position to express the arclength as a function of the curvature representing the quadrature (9) via elliptic integrals or elementary functions in each of the particular cases (I) and (II). Instead of this, however, taking the corresponding inverse functions we prefer to give directly the curvature as a function of the arclength in terms of the roots of the polynomial $P(\kappa)$. Moreover, solving the integral (7), we give explicit formulae for the corresponding slope angles as well. The explicit analytic expressions for the solutions of

equation (3) and the corresponding slope angles can be found in theorem 1, for case (I), and in theorem 2, for case (II).

Theorem 1. *Given μ, σ and E, let the roots α, β, γ and δ of the respective polynomial $P(\kappa)$ be as in case (I), that is $\alpha < \beta \in \mathbb{R}$, $\gamma, \delta \in \mathbb{C}$, $\delta = \bar{\gamma}$, and let $\eta \equiv (\gamma - \bar{\gamma})/2\mathrm{i}$. Then, except for the cases in which $(3\alpha + \beta)(\alpha + 3\beta) = \eta = 0$, the function*

$$\kappa_1(s) = \frac{(A\beta + B\alpha) - (A\beta - B\alpha)\mathrm{cn}(us, k)}{(A + B) - (A - B)\mathrm{cn}(us, k)} \tag{20}$$

of the real variable s, where

$$A = \sqrt{4\eta^2 + (3\alpha + \beta)^2}, \qquad B = \sqrt{4\eta^2 + (\alpha + 3\beta)^2}, \qquad u = \frac{1}{4}\sqrt{AB}, \tag{21}$$

$$k = \frac{1}{\sqrt{2}}\sqrt{1 - \frac{4\eta^2 + (3\alpha + \beta)(\alpha + 3\beta)}{\sqrt{[4\eta^2 + (3\alpha + \beta)(\alpha + 3\beta)]^2 + 16\eta^2(\beta - \alpha)^2}}}, \tag{22}$$

takes real values for each $s \in \mathbb{R}$ and satisfies equation (3). This function is periodic if and only if $\eta \neq 0$ or $\eta = 0$ and $(3\alpha + \beta)(\alpha + 3\beta) > 0$, its least period is $T_1 = (4/u)\mathrm{K}(k)$, and for $\sigma \neq 0$ its indefinite integral $\varphi_1(s)$ such that $\varphi_1(0) = 0$ is

$$\varphi_1(s) = \frac{A\beta - B\alpha}{A - B}s + \frac{(A + B)(\alpha - \beta)}{2u(A - B)}\Pi\left(-\frac{(A - B)^2}{4AB}, \mathrm{am}(us, k), k\right) + \frac{\alpha - \beta}{2u\sqrt{k^2 + \frac{(A-B)^2}{4AB}}}\arctan\left(\sqrt{k^2 + \frac{(A - B)^2}{4AB}}\frac{\mathrm{sn}(us, k)}{\mathrm{dn}(us, k)}\right). \tag{23}$$

In the cases in which $(3\alpha + \beta)(\alpha + 3\beta) = \eta = 0$, the function

$$\kappa_2(s) = \zeta - \frac{4\zeta}{1 + \zeta^2 s^2}, \tag{24}$$

where $\zeta = \alpha$ when $3\alpha + \beta = 0$ and $\zeta = \beta$ when $\alpha + 3\beta = 0$, satisfies equation (3) for each $s \in \mathbb{R}$ and its indefinite integral $\varphi_2(s)$ such that $\varphi_2(0) = 0$ reads

$$\varphi_2(s) = \zeta s - 4\arctan(\zeta s). \tag{25}$$

Proof. Let us begin with the simpler case $(3\alpha + \beta)(\alpha + 3\beta) = \eta = 0$. Here, on account of the definition of η and relations (16)–(19), a straightforward computation shows that the function (24) is the derivative of the function (25) and satisfies equation (3). The relation $\varphi_2(0) = 0$ is obvious.

Next, let us exclude the cases in which $(3\alpha + \beta)(\alpha + 3\beta) = \eta = 0$. Now, the condition $\alpha < \beta \in \mathbb{R}$, the definition of η and expressions (21) and (22) also imply that $\eta, A, B, u, k \in \mathbb{R}$, $AB \neq 0$, $u > 0$ and $0 \leqslant k \leqslant 1$. Hence, the function (20) is real-valued when $s \in \mathbb{R}$. Substituting the function (20) into equation (3) and taking into account relations (16)–(19), one can easily verify that the latter equation is satisfied. When $\eta \neq 0$, expression (22) implies $0 < k < 1$ and therefore the function (20) is periodic because the function $\mathrm{cn}(us, k)$ is periodic, with least period T_1. When $\eta = 0$ but $(3\alpha + \beta)(\alpha + 3\beta) \neq 0$ two alternative cases are to be considered, namely $(3\alpha + \beta)(\alpha + 3\beta) > 0$ and $(3\alpha + \beta)(\alpha + 3\beta) < 0$. In the first case, expression (22) leads to $k = 0$, which means that $\mathrm{cn}(us, k) = \cos(us)$ and hence the function (20) is periodic again. However, if $(3\alpha + \beta)(\alpha + 3\beta) < 0$, then expression (22) implies $k = 1$ meaning that $\mathrm{cn}(us, k) = \mathrm{sech}(us)$, and hence the function (20) is not periodic. Thus, having considered all possible cases, we may conclude that the function (20) is periodic if and only

if $\eta \neq 0$ or $\eta = 0$ and $(3\alpha + \beta)(\alpha + 3\beta) > 0$. Finally, differentiation of expressions (23) with respect to the variable s yields (20). The relation $\varphi_1(0) = 0$ is obvious. □

Theorem 2. *Given μ, σ and E, let the roots α, β, γ and δ of the respective polynomial $P(\kappa)$ be as in case (II), that is $\alpha < \beta < \gamma < \delta \in \mathbb{R}$. Consider the functions*

$$\kappa_3(s) = \delta - \frac{(\delta - \alpha)(\delta - \beta)}{(\delta - \beta) + (\beta - \alpha)\mathrm{sn}^2(us, k)}, \tag{26}$$

$$\kappa_4(s) = \beta + \frac{(\gamma - \beta)(\delta - \beta)}{(\delta - \beta) - (\delta - \gamma)\mathrm{sn}^2(us, k)}, \tag{27}$$

of the real variable s, where

$$u = \frac{1}{4}\sqrt{(\gamma - \alpha)(\delta - \beta)}, \qquad k = \sqrt{\frac{(\beta - \alpha)(\delta - \gamma)}{(\gamma - \alpha)(\delta - \beta)}}. \tag{28}$$

Then, both functions (26) and (27) take real values for each $s \in \mathbb{R}$ and satisfy equation (3), they are periodic with the least period $T_2 = (2/u)\,\mathrm{K}(k)$ and their indefinite integrals $\varphi_3(s)$ and $\varphi_4(s)$, respectively, such that $\varphi_3(0) = \varphi_4(0) = 0$ are

$$\varphi_3(s) = \delta s - \frac{\delta - \alpha}{u}\Pi\left(\frac{\beta - \alpha}{\beta - \delta}, \mathrm{am}(us, k), k\right), \tag{29}$$

$$\varphi_4(s) = \beta s - \frac{\beta - \gamma}{u}\Pi\left(\frac{\delta - \gamma}{\delta - \beta}, \mathrm{am}(us, k), k\right). \tag{30}$$

Proof. It is easy to see that the condition $\alpha < \beta < \gamma < \delta \in \mathbb{R}$ and expressions (28) also imply that $u, k \in \mathbb{R}, u > 0$ and $0 < k < 1$. Therefore, both functions (26) and (27) are real-valued when $s \in \mathbb{R}$. Substituting each of the above functions into equation (3) and taking into account relations (16)–(19), one can easily verify that they satisfy it. Evidently, these functions are periodic due to the fact that the function $\mathrm{sn}^2(us, k)$ is periodic, with least period $T_2 = (2/u)\,\mathrm{K}(k)$, since $u > 0$ and $0 < k < 1$. Finally, differentiation of expressions (29) and (30) with respect to the variable s yields (26) and (27), respectively. The relations $\varphi_3(0) = \varphi_4(0) = 0$ are obvious. □

Suppose now that $\sigma = 0$. Under this assumption, in case (I), formulae (16) and (18) imply $\beta = -\alpha > 0$. Then, according to theorem 1, $B = A = 2\sqrt{\eta^2 + \alpha^2}$ and so

$$\kappa_1(s) = \alpha\,\mathrm{cn}(us, k), \qquad u = \frac{1}{2}\sqrt{\eta^2 + \alpha^2}, \qquad k = \sqrt{\frac{\alpha^2}{\eta^2 + \alpha^2}}, \tag{31}$$

cf formulae (20)–(22). Consequently

$$\int \kappa_1^2(s)\,\mathrm{d}s = 2\sqrt{\alpha^2 + \eta^2}\,\mathrm{E}(\mathrm{am}(us, k), k) - \eta^2 s. \tag{32}$$

Note also that formulae (31) and (17) imply

$$\kappa_1(0) = \alpha, \qquad \mu = \tfrac{1}{2}(\alpha^2 - \eta^2). \tag{33}$$

In case (II), the assumption $\sigma = 0$ and formulae (16) and (18) imply $\delta = -\alpha > 0, \gamma = -\beta > 0$. Hence, according to theorem 2, cf formulae (26)–(28), we have

$$u = -\frac{1}{4}(\alpha + \beta) = \frac{1}{4}(\delta + \gamma), \qquad k = -\frac{\beta - \alpha}{\beta + \alpha} = \frac{\delta - \gamma}{\delta + \gamma},$$

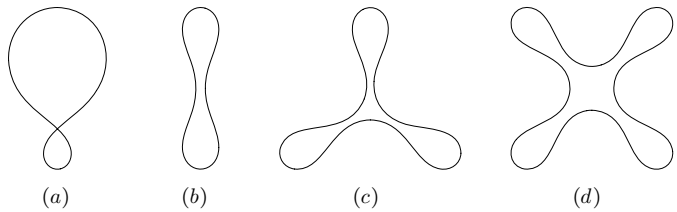

Figure 2. Directrices of some closed cylindrical equilibrium shapes whose curvatures are solutions to equation (3) of form (20) with $\sigma = 1$ and: (*a*) $\mu = 1.908$, $E = 0.146$; (*b*) $\mu = 0$, $E = 0.211$; (*c*) $\mu = 1/3$, $E = 0.407$ and (*d*) $\mu = 1/3$, $E = 0.563$.

and

$$\kappa_3(s) = \alpha \frac{1 - k\,\mathrm{sn}^2(us, k)}{1 + k\,\mathrm{sn}^2(us, k)}, \qquad \kappa_4(s) = \gamma \frac{1 + k\,\mathrm{sn}^2(us, k)}{1 - k\,\mathrm{sn}^2(us, k)}.$$

Then, using Gauss's transformation $\tilde{u} = u\,(1 + k)\,, \tilde{k} = 2\sqrt{k}/(k + 1)$, we obtain

$$\kappa_3(s) = \alpha\,\mathrm{dn}(\tilde{u}s, \tilde{k}), \qquad \tilde{u} = -\frac{1}{2}\alpha, \qquad \tilde{k} = -\frac{1}{\alpha}\sqrt{\alpha^2 - \beta^2}, \tag{34}$$

$$\kappa_4(s) = \gamma \frac{1}{\mathrm{dn}(\tilde{u}s, \tilde{k})} = -\alpha\,\mathrm{dn}(\tilde{u}s + \mathrm{K}(\tilde{k}), \tilde{k}) = -\kappa_3(s + \tilde{u}^{-1}\mathrm{K}(\tilde{k})). \tag{35}$$

Now, observing relation (35), we arrive at the conclusion that Euler's elastic curves of curvatures $\kappa_3(s)$ and $\kappa_4(s)$ determined by formulae (34) and (35), respectively, coincide up to a rigid motion in the plane $\mathbb{R}^2$. Therefore, to complete the present task it suffices to consider only the curves of curvature given by formula (34). In this case, we have

$$\int \kappa_3^2(s)\,\mathrm{d}s = -2\alpha\mathrm{E}(\mathrm{am}(\tilde{u}s, \tilde{k}), \tilde{k}) \tag{36}$$

and

$$\kappa_3(0) = \alpha, \qquad \mu = \tfrac{1}{2}(\alpha^2 + \beta^2). \tag{37}$$

due to relations (34) and (17).

Thus, having obtained in explicit form the solutions of equation (3), i.e., the curvatures, the corresponding slope angles and the integrals of the squared curvatures (in the cases in which $\sigma = 0$), we have completely determined in analytic form the corresponding directrices Γ (up to a rigid motion in the plane $\mathbb{R}^2$) through the parametric equations (11) when $\sigma \neq 0$ or (15), with the supplementary relations (31)–(33) or (34), (36) and (37), when $\sigma = 0$. Several examples of directrices of cylindrical equilibrium shapes corresponding to solutions to equation (3) of form (20), (26) or (27) are presented in figures 2 and 3 for $\sigma \neq 0$ and in figure 4 for $\sigma = 0$.

4. Closure conditions

Hereafter, we are interested in directrices Γ that close up smoothly meaning that there exists a value L of the arclength s such that $\mathbf{x}(0) = \mathbf{x}(L)$ and $\mathbf{t}(0) = \mathbf{t}(L)$. The later property of such a smooth closed directrix Γ and the definition of the tangent vector imply that there exists an integer m such that $\varphi(L) = \varphi(0) + 2m\pi$ where $\varphi(s)$ is the corresponding slope angle. Since

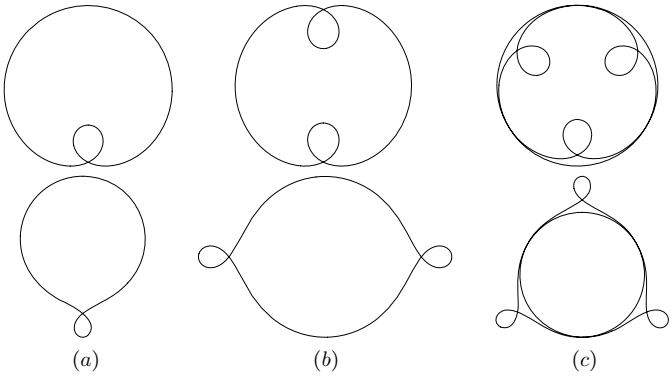

Figure 3. Directrices of some closed cylindrical equilibrium shapes whose curvatures are solutions to equation (3) of forms (26) (top) and (27) (bottom) with $\mu = 3$, $\sigma = 1$ and: (*a*) $E = 0.085\,0056$, (*b*) $E = 0.084\,9733$, (*c*) $E = 0.085\,0046$.

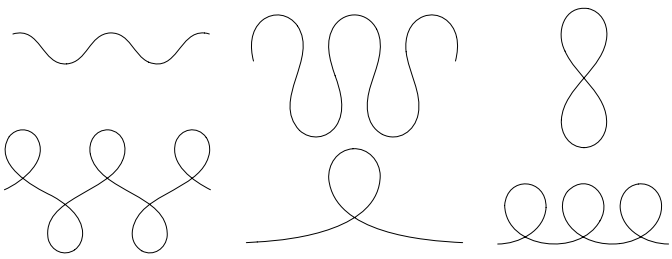

Figure 4. Several typical equilibrium shapes of Euler's elastica related to periodic and non-periodic solutions of equation (3) with $\sigma = 0$ of forms (31) and (34).

throughout this paper we always choose $\varphi(0) = 0$, this means that the condition $\mathbf{t}(0) = \mathbf{t}(L)$ implies $\varphi(L) = 2m\pi$.

Under the above assumptions, expressions (11) show that

$$\mathbf{x}(0) = \left(\frac{2}{\sigma}\left.\frac{\mathrm{d}\kappa(s)}{\mathrm{d}s}\right|_{s=0}, -\frac{1}{\sigma}(\kappa^2(0) - \mu)\right), \qquad \mathbf{x}(L) = \left(\frac{2}{\sigma}\left.\frac{\mathrm{d}\kappa(s)}{\mathrm{d}s}\right|_{s=L}, -\frac{1}{\sigma}(\kappa^2(L) - \mu)\right).$$

Consequently, the equality $\mathbf{x}(0) = \mathbf{x}(L)$ yields

$$\left.\frac{\mathrm{d}\kappa(s)}{\mathrm{d}s}\right|_{s=L} = \left.\frac{\mathrm{d}\kappa(s)}{\mathrm{d}s}\right|_{s=0}, \qquad \kappa(L) = \pm\kappa(0),$$

which, on account of relation (8), implies that L is a period of the curvature $\kappa(s)$, that is $L = nT$ where n is a positive integer and T is the least period of the function $\kappa(s)$. Since $\varphi(nT) = n\varphi(T)$, as follows by formula (7), then $2m\pi = \varphi(L) = \varphi(nT) = n\varphi(T)$ and hence

$$\varphi(T) = \frac{2m\pi}{n}. \tag{38}$$

Thus, in the cases when $\sigma \neq 0$, relation (38) is found to be a necessary condition for a directrix Γ to close up smoothly. Apparently, it is a sufficient condition as well.

Straightforward computations lead to the following explicit expressions

$$\varphi_1(T_1) = \frac{4(A\beta - B\alpha)}{u(A-B)}\mathrm{K}(k) + 2\frac{(A+B)(\alpha-\beta)}{u(A-B)}\Pi\left(-\frac{(A-B)^2}{4AB}, k\right),$$

$$\varphi_3(T_2) = \frac{2\delta}{u}\mathrm{K}(k) + 2\frac{\alpha-\delta}{u}\Pi\left(\frac{\alpha-\beta}{\delta-\beta}, k\right),$$
$$\varphi_4(T_2) = \frac{2\beta}{u}\mathrm{K}(k) + 2\frac{\gamma-\beta}{u}\Pi\left(\frac{\gamma-\delta}{\beta-\delta}, k\right),$$

for the angles of forms (23), (29) and (30), respectively. These expressions and the closure condition (38) allow to determine whether a curve of the curvature (20), (26) or (27) closes up smoothly or not.

In the cases when $\sigma = 0$, relations (14) and (15) show that

$$\left.\frac{\mathrm{d}\kappa(s)}{\mathrm{d}s}\right|_{s=L} = 0, \qquad \kappa(L) = \kappa(0), \qquad x(L) - x(0) = 0$$

are the necessary and sufficient conditions for the respective directrices Γ to close up smoothly. Let us recall that relations (14)–(15) were derived assuming $\varphi = \mathrm{d}\kappa/\mathrm{d}s = 0$ at $s = 0$. The aforementioned conditions mean first that $L = nT$ where n is a positive integer and T is the least period of the function $\kappa(s)$ and, consequently, that

$$\mu T = \int_0^T \kappa^2(s)\,\mathrm{d}s, \tag{39}$$

in view of the first of relations (15) and due to the fact that $\kappa(s)$ is a periodic function. Now, using formulae (32) and (36), one can easily see that

$$\int_0^{T_1} \kappa_1^2(s)\,\mathrm{d}s = 8\sqrt{\alpha^2+\eta^2}\,\mathrm{E}(k) - \eta^2 T_1, \qquad \int_0^{T_2} \kappa_3^2(s)\,\mathrm{d}s = -4\alpha\,\mathrm{E}(\tilde{k})$$

and then, expressing the coefficient μ from formulae (33) or (37), to rewrite the closure condition (39) for the curves of the curvatures given by formulae (31) or (34) in the form

$$2\mathrm{E}(k) - \mathrm{K}(k) = 0$$

or

$$2\mathrm{E}(\tilde{k}) - (2-\tilde{k}^2)\mathrm{K}(\tilde{k}) = 0,$$

respectively.

An interesting property of the curves of the curvatures $\kappa_3(s)$ and $\kappa_4(s)$ is observed in the cases in which $\sigma \neq 0$ (see figure 3). Namely, if one of these curves closes up smoothly, then so does the other one. To prove this let us first note that the solutions $\kappa_3(s)$ and $\kappa_4(s)$ correspond to case (II) when the polynomial $P(\kappa)$ has four real roots. Without loss of generality, these roots can be written in the form

$$\alpha = -3q - v - 2w, \quad \beta = q - v - 2w, \quad \gamma = q - v + 2w, \quad \delta = q + 3v + 2w, \tag{40}$$

where q, v and w are three arbitrary positive real numbers. The main advantage of this parametrization is that it preserves the order of the roots of the polynomial $P(\kappa)$, i.e., $\alpha < \beta < \gamma < \delta$ for any choice of the parameters q, v and w, which allows one to deal freely with them. Using these parameters, it is easy to find that

$$\frac{\partial\psi}{\partial q} = \frac{\partial\psi}{\partial v} = \frac{\partial\psi}{\partial w} = 0, \qquad \psi = \varphi_4(T_2) - \varphi_3(T_2),$$

meaning that $\psi = \text{const}$. This constant can be determined by evaluating the function ψ for any values of the parameters q, v and w, say $q = v = w$, which gives

$$\varphi_4(T_2) - \varphi_3(T_2) = 4\pi. \tag{41}$$

This relation and the closure condition (38) do imply that the foregoing two curves close up simultaneously.

5. Self-intersection

In what concerns the vesicle shapes, of special interest are solutions to equation (3) with $\sigma \neq 0$ that give rise to closed non-self-intersecting (simple) curves. A sufficient condition for such a closed curve to be simple is $\mu < 0$, which is discussed in [8]. It is also mentioned therein that the closed curves satisfying condition (38) with $m \neq \pm 1$ or $n = 1$ are necessarily self-intersecting. In this section, the case $\mu > 0$ is considered and several new sufficient conditions are presented for a closed curve of the foregoing type meeting a closure condition of form (38) with $m = \pm 1$ and $n \geqslant 2$ to be simple or not.

It is convenient to treat the problem of self-intersecting in terms of the magnitude $r(s)$ of the position vector $\mathbf{x}(s)$ and the angle $\theta(s)$ between the position vectors $\mathbf{x}(0)$ and $\mathbf{x}(s)$. Assuming that the angle $\theta(s)$ is positive when measured counterclockwise from the vector $\mathbf{x}(0)$ to the vector $\mathbf{x}(s)$ and negative otherwise and taking into account expressions (11) we obtain the relations

$$x(s) = -\mathrm{sgn}(z(0))r(s)\sin\theta(s), \qquad z(s) = \mathrm{sgn}(z(0))r(s)\cos\theta(s), \tag{42}$$

$$\frac{\mathrm{d}\theta(s)}{\mathrm{d}s} = \frac{\kappa^2(s) - \mu}{\sigma r^2(s)}. \tag{43}$$

The following observation is crucial for the rest of the present study.

Lemma 1. *Let Γ be a smooth closed curve whose curvature $\kappa(s)$ is a solution of equation (3) with $\sigma \neq 0$ of form (20), (26) or (27). Let T be the least period of the function $\kappa(s)$ and let the corresponding slope angle meet a closure condition of form (38) with $m = \pm 1$ and $n \geqslant 2$, i.e., $\varphi(T) = \pm 2\pi/n$. Then, the curve Γ is self-intersecting if and only if there exists $s_0 \in (0, T/2)$ such that $\theta(s_0) = \theta(0)$ or $\theta(s_0) = \theta(T/2)$.*

Proof. First, let us recall that each such curve has n axis of symmetry (cf [2, 8]), any position vector $\mathbf{x}(iT/2)$, $i = 0, 1, \ldots, 2n$ being along one of them. Consequently, any point of the curve whose position vector is collinear with one of the foregoing vectors lies on an axis of symmetry of the curve Γ. Note also that on account of formulae (23), (29) or (30) the closer condition may be written in the form $\varphi(T/2) = \pm\pi/n$. It should be mentioned as well that formulae (20), (26) and (27) imply $\mathrm{d}\kappa/\mathrm{d}s = 0$ at $s = \mathrm{i}T/2$.

Let the curve Γ be such that $\theta(s_0) = \theta(0)$ or $\theta(s_0) = \theta(T/2)$ at some $s_0 \in (0, T/2)$. Then, expressions (11) and (42), and the closer condition imply $\theta(T/2) = \pm(\pi/n + l\pi)$, where l is an integer, meaning that the position vector $\mathbf{x}(s_0)$ is along a certain axis of symmetry of the curve Γ. Therefore this curve self-intersects since it passes through two different points lying on one and the same axis of symmetry.

Next, suppose that the curve Γ is such that $\theta(s_0) \neq \theta(0)$ and $\theta(s_0) \neq \theta(T/2)$ for each $s_0 \in (0, T/2)$. Then, the same holds true in the next interval $(T/2, T)$ since the curve Γ is symmetric with respect to the axis corresponding to the angle $\theta(T/2)$, that is along the vector $\mathbf{x}(T/2)$, and so on up to the last interval $(nT - T/2, nT)$. In this way, we arrive at the conclusion that the considered curve does not pass twice through any one of its axes of symmetry and therefore it is simple because it is simple in the interior of each of the foregoing intervals too as is evident from relation (12). □

Theorem 3. *Let Γ be a smooth closed curve, which meets the assumptions of Lemma 1. Then:*

(i) the curve Γ is simple if $\kappa^2(s) - \mu \neq 0$ for $s \in [0, T/2]$;
(ii) the curve Γ is self-intersecting if the equation $\kappa^2(s) - \mu = 0$ has exactly one solution for $s \in [0, T/2]$.

Proof.

(i) The condition $\kappa^2(s) - \mu \neq 0$ and expression (43) imply $\mathrm{d}\theta(s)/\mathrm{d}s \neq 0$ for each $s \in [0, T/2]$, meaning that $\theta(s)$ is a strictly increasing or decreasing function in this interval. Consequently, there does not exist $s_0 \in (0, T/2)$ such that $\theta(s_0) = \theta(0)$ or $\theta(s_0) = \theta(T/2)$, and hence, by virtue of Lemma 1, the corresponding curve Γ is simple.

(ii) Let s_μ be the only value in $[0, T/2]$ such that $\kappa^2(s_\mu) - \mu = 0$. First, let $s_\mu \in (0, T/2)$. Since $\kappa(s)$ is strictly increasing in this interval, the signs of function $\kappa^2(s) - \mu$ in the intervals $(0, s_\mu)$ and $(s_\mu, T/2)$ are different. Then, expression (43) implies that the function $\theta(s)$ has an extremum at s_μ. Suppose, $\theta(s)$ has a maximum at s_μ. Then, when s increases from 0 to $T/2$, the function $\theta(s)$ increases from $\theta(0) = 0$ to a certain value $\theta(s_\mu) = \theta_{\max} > 0$, and after that, decreases from $\theta_{\max}$ to $\theta(T/2)$. If $\theta(T/2)$ is negative, then there exists $s_1 \in (s_\mu, T/2)$ such that $\theta(s_1) = \theta(0)$ meaning that the curve Γ is self-intersecting. If $\theta(T/2)$ is positive, then there exists $s_2 \in (0, s_\mu)$ such that $\theta(s_2) = \theta(T/2)$ meaning that the curve Γ is self-intersecting again. The case when $\theta(s)$ has a minimum at s_μ is similar. Next, let $s_\mu = 0$ or $s_\mu = T/2$. Then, expressions (11) imply $\mathbf{x}(0) = \mathbf{x}(T) = \mathbf{0}$ or $\mathbf{x}(T/2) = \mathbf{x}(3T/2) = \mathbf{0}$, respectively. On the other hand, $\mathbf{t}(0) \neq \mathbf{t}(T)$ in the first case because $\varphi(T) = 2\pi/n$, while in the second case $\mathbf{t}(T/2) \neq \mathbf{t}(3T/2)$ since $\varphi(T/2) = \pi/n$ and $\varphi(3T/2) = 3\pi/n$. Therefore, the corresponding curve Γ is self-intersecting. □

Corollary 1. *Under the assumptions of Lemma 1, the curve Γ is self-intersecting if its curvature $\kappa(s)$ is such that the respective polynomial $P(\kappa)$ has only real roots.*

Proof. In case (I), according to theorem 1, real roots are achieved only if $\eta = 0$ meaning that $\gamma = \delta = -(\alpha + \beta)/2$. Then, formula (17) implies $(\alpha^2 - \mu)(\beta^2 - \mu) < 0$ provided that $(3\alpha + \beta)(\alpha + 3\beta) > 0$ which is the necessary and sufficient condition for the periodicity of the curvature $\kappa_1(s)$ when $\eta = 0$ (see theorem 1). Consequently, the equation $\kappa_1^2(s) - \mu = 0$ has only one solution for $s \in [0, T/2]$ since $\kappa_1(0) = \alpha$ and $\kappa_1(T/2) = \beta$. Then, according to theorem 3 (*ii*), the curve is self-intersecting.

In case (II), the roots can be written in the form (40) and the angles $\varphi_3(T)$ and $\varphi_4(T)$ may be thought of as functions of the parameters q, v and w. Differentiating relation (41) with respect to the parameter w, one obtains

$$\frac{\mathrm{d}\varphi_3(T)}{\mathrm{d}w} = \frac{\mathrm{d}\varphi_4(T)}{\mathrm{d}w} = \frac{v^2 - q^2}{w(q + v + w)\sqrt{(q + w)(v + w)}}\mathrm{E}(k).$$

The function $\mathrm{E}(k) > 1$ since $0 < k < 1$ and therefore the above derivatives are positive when $v > q$, negative when $v < q$ and equal to zero when $v = q$. If $v > q$, then $\varphi_3(T)$ and $\varphi_4(T)$ are increasing functions of the variable w and meet the inequalities

$$\varphi_3(T) < -2\pi, \qquad \varphi_4(T) < 2\pi$$

since

$$\lim_{w\to\infty} \varphi_3(T) = -2\pi, \qquad \lim_{w\to\infty} \varphi_4(T) = 2\pi.$$

The first inequality implies that if the curve corresponding to the angle $\varphi_3(T)$ closes up, then it necessarily self-intersects since $m < -1$ in the respective closure condition (38). Consider now the curve, corresponding to $\varphi_4(T)$. It is easy to see that

$$(\gamma^2 - \mu)(\delta^2 - \mu) = 4[(q + v)^2 + 4vw][q^2 - 3v^2 - 2qv - 4vw] < 0$$

since in this case $q^2 - 3v^2 < 0$. Consequently, the equation $\kappa_4^2(s) - \mu = 0$ has only one solution for $s \in [0, T/2]$ since $\kappa_4(0) = \gamma$ and $\kappa_4(T/2) = \delta$. Therefore, according to theorem

3 (*ii*), the curve corresponding to $\varphi_4(T)$ is self-intersecting. Thus, in the case $v > q$ both curves are self-intersecting. Similar arguments hold in the case $v < q$ and lead to the same conclusion. Finally, if $v = q$, then the functions $\varphi_3(T)$ and $\varphi_4(T)$ do not depend on w. Evaluating them at $q = w$, one obtains

$$\varphi_3(T) = -2\pi, \qquad \varphi_4(T) = 2\pi$$

and hence the two curves are self-intersecting as well. □

6. Concluding remarks

In this paper all solutions to the membrane shape equation (2) determining cylindrical surfaces are presented in analytic form in theorems 1 and 2. Explicit analytic expressions for the corresponding slope angles in the case $\sigma \neq 0$ are also given in these theorems. Explicit expressions for the integrals of the squared curvatures are obtained in the case $\sigma = 0$. In this way, we have completely determined in analytic form the corresponding directrices Γ (up to a rigid motion in the plane $\mathbb{R}^2$) through the parametric equations (11) or (15) depending on whether $\sigma \neq 0$ or $\sigma = 0$. The necessary and sufficient conditions for the foregoing cylindrical surfaces to be closed are derived in section 4. One necessary and sufficient and three sufficient conditions for their directrices to be simple or not are presented in section 5. These conditions show that simple directrices could be achieved only in case (I), i.e., when the polynomial $P(\kappa)$ has two simple real roots and a pair of complex conjugate roots.

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